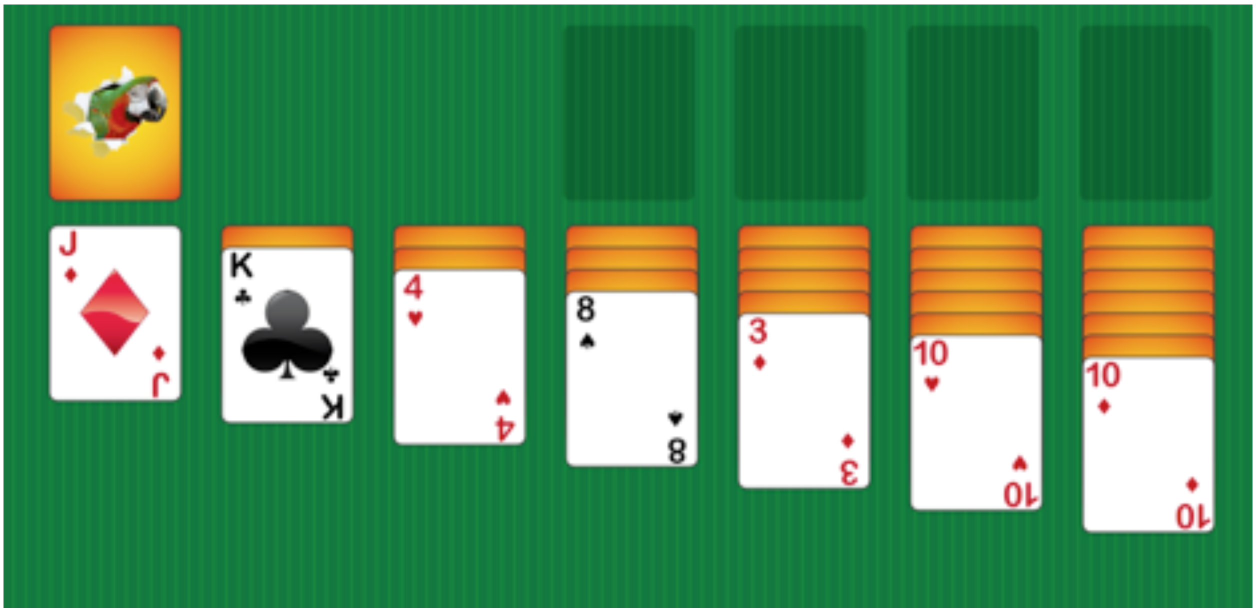
Probabilities of Different Combinations of Cards Showing in the Initial Layout for Klondike Solitaire

B. A. Hughes
2017 November 16



A. Introduction

In the initial layout of Klondike Solitaire there are patterns in the 7 turned-up cards that can be interpreted as an indication of ineffective randomization of the cards before the layout, or of some artifact leading to unexpected correlations occurring in the order of the cards. In order to provide a basis for deciding whether this interpretation is correct or not, the present analysis determines the mathematical frequencies of a variety of patterns assuming no inherent order before the layouts. The patterns chosen for analysis are in 4 categories and are the following: 11 kinds of multiples within ranks (pairs, threes-of-a-kind,…), 7 kinds of straight distributions, 4 kinds of colour distributions, and 6 kinds of flush distributions — in all, a total of 28 patterns. In the mathematical analysis, it will be necessary to use the total number of ways that 7 cards can be chosen from 52 without regard for their order, which is 52C7 (133,764,560), where nCm is the binomial coefficient $n!/[m!\,(n-m)!]$. For each of the four categories the number of kinds make a complete set, so that the sum of probabilities of the kinds in each category is unity, and the total number of ways all the kinds of patterns in a given category can appear is 52C7.

In addition, one particular computer-based solitaire game — Growly Solitaire[1] — has been used to provide hundreds of trials for comparison with the mathematical expectations.

B. Definition of the various patterns

1. MULTIPLES WITHIN RANKS

Let the probability of each of the 11 kinds *showing by themselve*s be designated by P_{ijk} where each subscript indicates the number of cards in a rank (e.g. 2 for a pair, 3 for a 3-of-a-kind, etc.). Each must be showing on its own, so that a full house, for example, doesn't count as a pair or a 3-or-a-kind, but only for the single item "3-of-a-kind and a pair". Each multiple within a rank may be accompanied by any distribution of straights or colours. The different kinds are as follows:

P_1 .... All cards with different ranks
P_2 .... One pair
P_{22} .. Two pairs
P_{222}. Three pairs
P_3 .... One 3-of-a-kind (triple)
P_{32} .. One 3-of-a-kind and a pair (full house)
P_{322} .... One 3-of-a-kind and 2 pairs
P_{33} ...... Two 3's-of-a-kind
P_4 .... One 4-of-a-kind (quadruple)
P_{42} ...... One 4-of-a-kind and one pair
P_{43} ...... One 4-of-a-kind and one 3-of-a-kind

In the above image, there is one pair (10s) and it would count towards P_2.

2, STRAIGHTS

Let the probability of each of the 7 kinds be designated by P_{Si} where the number-subscript indicates the number of cards in an item (e.g. 3 for a 3-rank straight, 4 for a 4-rank straight, etc.). Each must be the longest straight showing, so that a 3-rank straight may show two 3-rank straights and one or two 2-rank straights as well, but no 4-rank straights or longer. Each kind of straight combination may have with it any distribution of rank multiples such as a pair, 2 pairs, 3-of-a-kind, etc., and any distribution of colour. The different kinds are as follows:

P_{S1} .... all cards from non-adjacent ranks
P_{S2} .... 2-rank straights
P_{S3} .... 3-rank straights
P_{S4} .... 4-rank straights
P_{S5} ..... 5-rank straights
P_{S6} ..... 6-rank straights
P_{S7} ..... 7-rank straights

In the above image, there are two 2-rank straights, (3-4, and 10-J which appears twice). This layout regarding straights would be given a count of one and it be included in the count for P_{S2}.

3. COLOUR

Let the probability of each of the 4 kinds showing by themselves be designated by P_{Ci} where i indicates the number of cards with the same colour (and $7 - i$ with the other colour). Straights and ranks have no special merit:

P_{C4} ......4 cards of one colour
P_{C5} ......5 cards of one colour
P_{C6} .....6 cards of one colour
P_{C7} .....7 cards of one colour

The image above has a maximum of 5 cards of one colour (red) and would count towards P_{C5}.

4. FLUSH

Let the probability of each of the 6 kinds showing regardless of the other 3 cards except that they must be in different suits from the chosen one be designated by P_{Fi} where i indicates the number of cards with the same suit (and $7 - i$ with the other suits). Again, straights and ranks have no special merit:

P_{F2} .....maximum of 2 cards in one suit (i.e. 2221 as the number of cards in each of the suits)
P_{F3} .....maximum of 3 cards in one suit (i.e. 3211, 3220, 3310)
P_{F4} .....4 cards in one suit (i.e. 4111, 4210, 4300)
P_{F5} ..... 5 cards in one suit (i.e. 5110, 5200)
P_{F6} ..... 6 cards in one suit (i.e. 6100)
P_{F7} ..... 7 cards in one suit

The image above has a 3-card flush as its maximum and its count of one would go towards P_{F3}.

C. Theoretical Probabilities

The integer in parentheses after the value of the probability is the value of the numerator, i.e. the total 'count' for that case.

1. MULTIPLES WITHIN RANKS

P_1: There are 7 different ranks in the final grouping. The number of ways 7 cards of different ranks can be chosen from the 13 ranks is 13C7 with each rank multiplied by 4 for the number of suit possibilities:

$$P_1 = \frac{13C7 \bullet 4^7}{52C7} = 0.21015088 \qquad (28{,}114{,}944)$$

P_2: There are 6 different ranks in the final grouping. The number of ways 6 cards of different ranks can be chosen from the 13 ranks is 13C6 with 5 of them — the cards that will remain unmatched — multiplied by 4 for the number of suits possible, and the total multiplied by 6 for the number of cards that can be matched up and finally multiplied by 6, the number of ways of choosing a pair from the 4 suits in the matched-up card:

$$P_2 = \frac{13C6 \bullet 4^5 \bullet 6 \bullet 6}{52C7} = 0.472839497 \qquad (63{,}258{,}624)$$

P_{22}: There are 5 different ranks in the final grouping. The number of ways 5 cards of different ranks can be chosen from the 13 ranks is 13C5 with 3 of them — the cards that will remain unmatched — multiplied by 4 for the number of suits possible, and the total multiplied by the number of ways 2 ranks can be chosen from the 5 different ranks (5C2, to make the pairs) and with 6 each for the number of ways a pair can be drawn from the 4 cards in a suit:

$$P_{22} = \frac{13C5 \bullet 4^3 \bullet 5C2 \bullet 6 \bullet 6}{52C7} = 0.221643514 \qquad (29{,}652{,}480)$$

P_{222}: There are 4 different ranks in the final grouping. The number of ways 4 cards of different ranks can be chosen from the 13 ranks is 13C4 with 1 of them — the card that will remain unmatched — multiplied by 4 for the number of suits possible, and the total multiplied by 4C3 for the number of ways 3 ranks can be drawn from 4, and then multiplied by 6^3 for number of ways each of the pairs can be chosen from their respective 4 suits:

$$P_{222} = \frac{13C4 \bullet 4 \bullet 4C3 \bullet 6 \bullet 6 \bullet 6}{52C7} = 0.01847029 \qquad (2{,}471{,}040)$$

P_3: There are 5 different ranks in the final grouping. The number of ways 5 cards of different ranks can be chosen from the 13 ranks is 13C5 with 4 of them — the cards that will remain unmatched — multiplied by 4 for the number of suits possible, and the total multiplied by 5 for the number of cards that can be matched up and finally multiplied by 4 for the number of ways of choosing 3-of-a-kind from the 4 suits in the matched-up card:

$$P_3 = \frac{13C5 \bullet 4^4 \bullet 5 \bullet 4}{52C7} = 0.049254114 \qquad (6{,}589{,}440)$$

P_{32}: There are 4 different ranks in the final grouping. The number of ways 4 cards of different ranks can be chosen from the 13 ranks is 13C4 with 2 of them — the cards that will remain unmatched — multiplied by 4 for the number of suits possible, and the total multiplied by 4 for the number of ranks possible for the triple multiplied by 4 for the number of ways a triple can be chosen from the 4 suits, and then multiplied by 3 for the number of ranks possible for the pair and finally multiplied by 6 for the number of pairs possible in the 4 suits for the pair:

$$P_{32} = \frac{13C4 \bullet 4^2 \bullet 4 \bullet 4 \bullet 3 \bullet 6}{52C7} = 0.024627057 \qquad (3{,}294{,}720)$$

P_{322}: There are 3 different ranks in the final grouping. The number of ways 3 ranks can be chosen from the 13 ranks is 13C3 multiplied by 3 for the number of ways the 3 ranks can be rearranged amongst themselves (given that 2 are threes-of-a-kind and so are not to be distinguished), and then by 4 for the number of ways 3 cards can be drawn from 4 for the triple, and multiplied by 6^2 for number of ways each of the pairs can be chosen from their respective 4 suits:

$$P_{322} = \frac{13C3 \bullet 3 \bullet 4 \bullet 6 \bullet 6}{52C7} = 0.000923515 \qquad (123{,}552)$$

P_{33}: There are 3 different ranks in the final grouping. The number of ways 3 different ranks can be chosen from the 13 ranks is 13C3 with 1 of them — the card that will remain unmatched — multiplied by 4 for the number of suits possible, and the total multiplied by 3C2 for the number of ways 2 ranks can be chosen from the 3 ranks (for the 2 triples), and then multiplied by 4^2 for the number of ways of choosing a triple from the 4 suits for each of the two triples:

$$P_{33} = \frac{13C3 \bullet 4 \bullet 3C2 \bullet 4 \bullet 4}{52C7} = 0.000410451 \qquad (54{,}912)$$

P_4: There are 4 different ranks in the final grouping. The number of ways 4 different ranks can be chosen from the 13 ranks is 13C4 with 3 of them — the cards that will remain unmatched — each multiplied by 4 for the number of suits possible, and the total multiplied by 4 for the number of ways the 4 suits can be arranged amongst themselves given that 3 ranks have 1 card only and so rearranging them makes no difference to the result (there is only 1 way for the quadruple regarding any inner arrangement):

$$P_4 = \frac{13C4 \bullet 4^3 \bullet 4}{52C7} = 0.001368170 \qquad (183{,}040)$$

P_{42}: There are 3 different ranks in the final grouping. The number of ways 3 cards of different ranks can be chosen from the 13 ranks is 13C3 with that multiplied by 6 for the number of ways the 3 ranks (they are all different) can be arranged amongst themselves, and by 6 for the number of ways of choosing a pair from 4 suits, and 4 for the number of choices amongst the 4 suits for the remaining card (there is only 1 way for the quadruple):

$$P_{42} = \frac{13C3 \bullet 3 \bullet 2 \bullet 6 \bullet 4}{52C7} = 0.000307838 \qquad (41{,}184)$$

P_{43}: There are 2 different ranks in the final grouping. The number of ways 2 ranks can be chosen from the 13 ranks is 13C2 with that multiplied by 2 for the number of ways the ranks can be arranged (the 4-card rank having a higher rank-number than the 3-card rank or vice versa), and then by 4 for the number of ways of choosing a triple from the 4 suits in the 3-card rank:

$$P_{43} = \frac{13C2 \bullet 2 \bullet 4}{52C7} = 0.000004664 \qquad (624)$$

The sum of the 11 numerators is exactly 52C7 so the probabilities sum to exactly 1.

2. STRAIGHTS

For all straights considered here, Ace is fixed as a low card — the run A,2,3,4,5 is a 5-card straight whereas 10,J,Q,K,A is a 4-card king-high straight. In the poker community (and perhaps in other games) Ace can be either low or high[2,3]. The present positioning for Ace follows the usual practice for Klondike solitaire.

There are two theoretical approaches that have been considered. The first is to compute all possible combinations that give rise to a particular straight run length, and subtract the extraneous ones from them. For a straight of run length 2, all run lengths of 3, 4 and 5 which would be included in the number of combinations of runs of length 2, would need to be determined and subtracted off, and for a straight run of length 3, it would mean subtracting off all straights with run length of 4. The other approach — which is the one adopted here — is to compute combinations that include only runs of a given length to begin with, even though there may be many sub-cases involved.

Although there are only 7 straights that are finally considered, there are a very large number of sub-cases that contribute to the probabilities. Each will be described and determined in turn.

Also in this category, a special counting circumstance arises because of the need to have non-adjacent ranks "buffering" the straights in question. This requires either extra ranks to be included in the counting, or a reduction in the overall number of ranks from which the straight pattern including the extra ranks can be chosen. A straightforward counting procedure including all the extra ranks results in multi-nested sums of sums of the natural numbers, a quite complicated process. In this present treatise the procedure instead will be to reduce the number of ranks from 13 from which the relevant ranks can be chosen. The reduction number is the extra number of ranks in the non-single rank entries (2-rank straights, 3-rank straights…) and 1 each for the "buffer" zones separating the non-adjacent ranks (1 for a zone rather than each rank). This allows the determination of the number of choices to be made by the combinatorial factor nCm where n is the number of reduced ranks (reduced from 13) and m is the number of elements in the combination — an element being a separate rank or a straight of any length (counting only 1 for each). This is shown in the following diagram where rows 2 to 3 and 5 to 7 are straights, and rows 9 and 12 are separate ranks. Note that ranks 10 and 11 constitute one "buffer" zone. In this case the number of ways the depicted ranks can be arranged is 7C4 (i.e. choosing 4 active items out of 7 active items) as is shown in the reduced depiction:

Ranks	Ranks to be deleted in the counting
13	
12	
11	
10	
9	
8	
7	
6	
5	
4	
3	
2	
1	

7
6
5
4
3
2
1

A mathematical identity arises out of the two ways of doing the counting. The first way of multi-nested sums can be written as

$$Number\ of\ ways = \sum_{N_{m-1}=1}^{N_m} \sum_{N_{m-2}=1}^{N_{m-1}} \sum_{N_{m-3}=1}^{N_{m-2}} \dots \sum_{N_1=1}^{N_2} \sum_{i=1}^{N_1} i$$

and the identity is therefore

$${}_{m-1+N_m}C_m = \sum_{N_{m-1}=1}^{N_m} \sum_{N_{m-2}=1}^{N_{m-1}} \sum_{N_{m-3}=1}^{N_{m-2}} \dots \sum_{N_1=1}^{N_2} \sum_{i=1}^{N_1} i\,.$$

As an example, for the case of 9C4, the identity becomes

$$9C4 = \sum_{q=1}^{6}\sum_{p=1}^{q}\sum_{n=1}^{p}\sum_{i=1}^{n} i \ .$$

The determination of the probabilities for each of the 7 straights can be carried out as follows:

P_{S1}: For no straights at all, i.e. the ranks of all 7 cards are non-adjacent, there are 6 variations — all 7 cards in either 2 ranks, or 3 ranks, or 4 ranks, … or 7 ranks. For all cases, each rank must be separated from the other ranks by at least one non-contributing rank, so that 2 ranks actually comprise at least 3 ranks, 3 comprise at least 5, 4 comprise at least 7,,,, and 7 ranks comprise all 13 ranks. The number of ways each can occur are determined as follow:

2 ranks: There are 4 cards in one rank and 3 in the other and that gives 4 ways the suits can be chosen. Using the reduction process described above there are 12C2 (66) different ways they can be fitted into the 13 ranks, and 2 ways they can be arranged between themselves. The total number of ways is the product 66•2•4 which is 528.

3 ranks: There are 3 ways the cards can be distributed amongst the 3 ranks — 3 in one rank, 3 in another rank and 1 in the remaining rank, 331, the second arrangement is 421, and the third is 322. There are 165 different ways they can be fitted into the 13 ranks (11C3), 3 ways they can be arranged amongst the 3 ranks for 331 and 322, and 6 ways for 421. The number of ways the suits can be arranged is 4^3 for 331, 6•4 for 421 and 4•6^2 for 322. The total number of ways these can be arranged is 165•(3•4^3+6•6•4+3•4•6^2), which is 126,720.

4 ranks: There are 3 ways the cards can be distributed amongst the 4 ranks — 2221, 3211, and 4111. There are 210 different ways they can be fitted into the 13 ranks (10C4), 4 ways they can be arranged amongst the 4 ranks for 2221 and 4111, and 12 ways for 3211. The suits can be arranged in 6^3•4 ways for 2221, 4^3•6 ways for 3211, and 4^3 ways for 4111. The total number of ways these can be arranged is 210•(4•6^3•4+12•4^3•6+4•4^3), which is 1,747,200

5 ranks: There are 2 ways the cards can be distributed amongst the 5 ranks — 31111, and 22111. There are 126 different ways they can be fitted into the 13 ranks (9C5), 5 ways they can be arranged amongst the 4 ranks for 31111, and 10 ways for 22111. The suits can be arranged in 4^5 ways for 31111, and 4^3•6^2 ways for 22111. The total number of ways these can be arranged is 126•(5•4^5+10•4^3•6^2), which is 3,548,160.

6 ranks: There is 1 way the cards can be distributed amongst the 6 ranks — 211111. There are 28 different ways they can be fitted into the 13 ranks (8C6), and there are 6 ways they can be arranged amongst the 4 ranks. The

suits can be arranged in 4^5•6 ways. The total number of ways these can be arranged is 28•(6•4^5•6), which is 1,032,192.

7 ranks: There is 1 way the cards can be distributed amongst the 7 ranks — 1111111. There is 1 way they can be fitted into the 13 ranks, and 4^7 ways the suits can be arranged. The total number of ways these can be arranged is 4^7, which is 16,384.

The probability is the sum of all the ways divided by 52C7:

$$P_{S1} = 0.0487370186 \qquad (6{,}471{,}184)$$

P_{S2}: For all cases a 2-rank straight is included. There are 6 variations — the 2-rank straight comprising only the 2 cards, the 2 ranks comprising 3 cards (i.e. one of the cards is paired up), the straight with 4 cards, 5 cards, 6 cards or 7 cards. Some of these variations have more than one internal arrangement, and more than one arrangement of the cards outside the ranks that make up the straight. Each will be dealt with in turn.

<u>7 cards within the 2 ranks of the straight</u> (no cards outside those ranks): There are 4 cards in one of the straight's ranks and 3 in the other. There are 12 ways the straight can be fitted into the 13 ranks, 2 arrangements of the ranks within the straight, and 4 ways the suits can be arranged. The total number of ways is 12•2•4, which is 96.

<u>6 cards within the 2 ranks of the straight</u> (1 card outside those ranks): There are 2 possible arrangements of the 6 cards, 3 in one rank and 3 in the other (3,3) and also a 4,2. There is one external rank with one card in it. There are 55 ways the straight and the external card can be fitted into the 13 ranks (11C2), and there 2 arrangements of the external rank and the straight. Within the straight, there are 4^2 suit arrangements of 3,3, and 6•2 suit arrangements of 4,2 (the '6' pertains to the '2' in 4,2 and the '2' pertains to the fact that it can be 4,2 or 2,4. There are 4 suit arrangements in the external rank. The total number of ways is 55•2•(6•2+4^2)•4, which is 12,320.

<u>5 cards within the 2 ranks of the straight</u> (2 cards outside those ranks in a single rank or 2 non-adjacent ranks or as a 2-rank straight): There are 2 possible arrangements of the 5 cards, 3 in one rank and 2 in the other (3,2) or a (4,1). For the case of one external rank with 2 cards in it, there are 55 ways the straight and the external rank can be fitted into the 13 ranks (11C2), and there are 2 arrangements of the external rank and the straight. Within the straight, there are 4•6•2 suit arrangements of 32, and 4•2 suit arrangements of 41. There are 6 suit arrangements in the external rank. The number of ways for this case is 55•2•6•(4•6•2+4•2), which is 36,960. For the case of 2 external non-adjacent ranks, there is one card in each, and 120•3 ways to fit and arrange the straight and 2 ranks in the 13 ranks (10C3•3). There are 4^2 suit arrangements for the 2 external ranks, giving a total of 120•3•(4•6•2+4•2)•4^2, which is 322,560. The remaining case of the external 2-rank straight (with 1 card in each rank) provides 45•2 ways to arrange and fit the 2 straights in the 13 ranks (10C2•2), with the other arrangements as in

the 2 external ranks, i.e. 45•2•(4•6•2+4•2)•4^2, or 80,640. The overall total of these 3 cases is 440,160.

4 cards within the 2 ranks of the straight and 3 cards outside those ranks — in a single rank, 2 non-adjacent ranks, 3 non-adjacent ranks, a 2-rank straight containing 3 cards, or a 2-rank straight and a separate non-adjacent rank: For these 5 cases the number of ways the 2-rank straight and the external ranks can be fitted in the 13 ranks are 11C2, 10C3, 9C4, 10C2, 9C3 respectively. The 4 cards can be arranged within the 2-rank straight in 2 ways — 2,2 and 3,1 — and this results in 10 separate cases. The arrangements of the 2-rank straight and the outside ranks for the 10 cases occur in the following number of ways, with the 2,2 straight listed for the first 5 and the 3,1 for the second: 2, 6, 10, 2, 6; 2, 6, 10, 2,6. The suit arrangements for each of the 10 cases is as follows, again, with the 2,2 straight listed first: 6^2•4, 6^2•6•4•2, 6^2•4^3, 6^2•6•4•2, 6^2•4^3; 4^2•2•4, 4^2•2•6•4•2, 4^2•2•4^3, 4^2•2•6•4•2, 4^2•2•4^3. All 10 cases are summarized in the following table.

2-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
2,2 & 1 rank (3)	11C2 (55)	2	6^2•4	15,840
2,2 & 2 ranks (2,1)	10C3 (120)	6	6^2•6•4	622,080
2,2 & 3 ranks (111)	9C4 (126)	4	6^2•4^3	1,161,216
2,2 & 2-rank straight (2,1)	10C2 (45)	2	6^2•6•4•2	155,520
2,2 & 2-rank str.(1,1) & 1 rank (1)	9C3 (84)	6	6^2•4^3	1,161,216
3,1 & 1 rank (3)	11C2 (55)	2	4^2•2•4	14,080
3,1 & 2 ranks (2,1)	10C3 (120)	6	4^2•2•6•4	552,960
3,1 & 3 ranks (111)	9C4 (126)	4	4^2•2•4^3	1,032,192
3,1 & 2-rank straight (2,1)	10C2 (45)	2	4^2•2•6•4•2	138,240
3,1 & 2-rank str.(1,1) & 1 rank (1)	9C3 (84)	6	4^2•2•4^3	1,032,192
			Total	**5,885,536**

3 cards within the 2 ranks of the straight and 4 cards outside those ranks — in a single rank, 2 non-adjacent ranks (2,2), 2 non-adjacent ranks (3,1), 3 non-adjacent ranks, 4 non-adjacent ranks, a 2-rank straight containing 4 cards (2,2), a 2-rank straight containing 4 cards (3,1), a 2-rank straight containing 3 cards and a separate rank, a 2-rank straight containing 2 cards and a separate rank, a 2-rank straight containing 2 cards and 2 separate ranks, and 2 2-rank straights. For these 10 cases the number of ways the 2-rank straight and the external ranks can be fitted in the 13 ranks are 11C2, 10C3, 10C3, 9C4, 8C5, 10C2, 9C3, 8C4, and 8C3 respectively. The 4 cards can be arranged within the 2-rank straight only as 2,1 but with 2 ways that this can be arranged (2,1 or 1,2). The arrangements of the 2-rank straight and the outside ranks for the 11 cases occur in the following number of ways: 2, 3, 6, 12, 5, 2, 2, 3, 6,12, 3. The suit arrangements for each of the 11 cases is as follows: 6•4•2, 6^3•4•2, 6•2•4^3, 6^2•4^3•2•3, 6•4^5•2•5, 4•6^3•2•3, 4^3•6•2•2, 6^2•2•2•4^3, 6^2•4^3•2, 6•4^5•2 and 6•4^5•2. All 11 cases are summarized in the following table.

2-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
2,1 & 1 rank (4)	11C2 (55)	2	6•4•2	5,280
2,1 & 2 ranks (2,2)	10C3 (120)	3	6^3•4•2	622,080
2,1 & 2 ranks (3,1)	10C3 (120)	6	6•2•4^3•2	552,960
2,1 & 3 ranks (2,1,1)	9C4 (126)	12	6^2•4^3•2	6,967,296
2,1 & 4 ranks (1111)	8C5 (56)	5	6•4^5•2	3,440,640
2,1 & 2-rank straight (2,2)*	10C2 (45)	2	4•6^3•2	—
2,1 & 2-rank straight (3,1)*	10C2 (45)	2	4^3•6•2•2	—
2,1 & 2-rank str.(2,1) & 1 rank (1)	9C3 (84)	3	6^2•2•2•4^3	2,322,432
2,1 & 2-rank str.(1,1) & 1 rank (2)	9C3 (84)	6	6^2•4^3•2	2,322,432
2,1 & 2-rank str.(1,1) & 2 ranks (1,1)	8C4 (70)	12	6•4^5•2	10,321,920
2,1 & 2 2-rank strs.(1,1) (1,1)	8C3 (56)	3	6•4^5•2	2,064,384
			Total	**28,619,424**

*these combinations are counted in "3 cards within the 2-ranks…." in the cases "2,2 & 2-rank straight (2,1)" and "3,1 & 2-rank straight (2,1)"

<u>2 cards within the 2 ranks of the straight</u> and 5 cards outside those ranks — in 2 non-adjacent ranks (3,2) and (4,1), 3 non-adjacent ranks (221) and (311), 4 non-adjacent ranks (2111), and 5 non-adjacent ranks (11111), a 2-rank straight containing 5 cards (3,2) and (4,1), a 2-rank straight containing 4 cards (2,2) and (3,1) and each of these with 1 separate rank (1), a 2-rank straight containing 3 cards (2,1) and 1 rank containing 2 cards, a 2-rank straight containing 3 cards (2,1) and 2 non-adjacent ranks (1,1) and a 2-rank straight containing 3 cards (2,1) and a 2 card straight (1,1), a 2-rank straight containing 2 cards and a separate rank (3), a 2-rank straight containing 2 cards and 2 separate ranks (2,1), a 2-rank straight containing 2 cards (1,1) and3 separate ranks with 1 card each, 2 2-rank straights (2,1 (1,1) and 2 2-rank straights (1,1) each with a separate rank. For these 17 cases the results are summarized in the following table.

The sum of the "Totals" for all 6 variations divided by 52C7 is the probability of the initial layout showing a 2-rank straight as the longest run of adjacent ranks,

$$P_{S2} = \frac{96 + 12{,}320 + 440{,}160 + 5{,}885{,}536 + 28{,}774{,}944 + 24{,}077{,}312}{52C7} = 0.441268021 \text{ (59,034,848)}$$

2-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
1,1 & 2 ranks (3) (2)	10C3 (120)	6	6•4^3	276,480
1,1 & 2 ranks (4) (1)	10C3 (120)	6	4^3	46,080
1,1 & 3 ranks (2) (2) (1)	9C4 (126)	12	6^2•4^3	3,483,648
1,1 & 3 ranks (3) (1) (1)	9C4 (126)	12	4^5	1,548,288
1,1 & 4 ranks (2) (1) (1) (1)	8C5 (56)	20	6•4^5	6,881.280
1,1 & 5 ranks (1) (1) (1) (1) (1)	7C6 (7)	6	4^7	688,128
1,1 & 2-rank straight (3,2)*	10C2 (45)	2	6•4^3•2	—
1,1 & 2-rank straight (4,1)*	10C2 (45)	2	4^3•2	—
1,1 & 2-rank str.(2,2) & 1 rank (1)**	9C3 (84)	6	6^2•4^3	—
1,1 & 2-rank str.(3,1) & 1 rank (1)**	9C3 (84)	6	4^5•2	—
1,1 & 2-rank str.(2,1) & 1 rank (2)***	9C3 (84)	6	6^2•4^3•2	—
1,1 & 2-rank str.(1,1) & 1 rank (3)	9C3 (84)	3	4^5	258,048

1,1 & 2-rank str.(2,1) & 2 ranks (1) (1)[†]	8C4 (70)	12	6•4^5•2	—
1,1 & 2-rank str.(1,1) & 2 ranks (1) (2)	8C4 (70)	12	6•4^5	5,160,960
1,1 & 2-rank str.(1,1) & 3 ranks (1) (1) (1)	7C5 (21)	10	4^7	3,440.640
1,1 & 2 2-rank strs.(1,1) (1) (2)[††]	8C3 (56)	3	6•4^5•2	—
1,1 & 2 2-rank strs.(1,1) (1,1) & 1 rank (1)	7C4 (35)	4	4^7	2,293,760
			Total	**24,077,312**

*these combinations are counted in "5 cards within the 2 ranks……" in the case of the "external 2-rank straight"

** these combinations are counted in "4 cards within the 2 ranks……" in the cases of "2,2 & 2-rank str.(1,1) & 1 rank (1)" and "3,1 & 2-rank str.(1,1) & 1 rank (1)"

*** this combination is counted in "3 cards within the 2 ranks….." in the case of "2,1 & 2-rank str.(1,1) & 1 rank (2)"

[†]this combination is counted in "3 cards within the 2 ranks….." in the case of "2,1 & 2-rank str.(1,1) & 2 ranks (1) (1)"

[††]this combination is counted in "3 cards within the 2 ranks….." in the case of "2,1 & 2 2-rank strs.(1,1) (1,1)"

P_{S3}: For all cases a 3-rank straight is included. There are 5 variations — the 3-rank straight comprising only the 3 cards, the straight comprising 4 cards (i.e. one of the cards is paired up), the straight with 5 cards, 6 cards or 7 cards. As for 2-rank straights some of these variations have more than one internal arrangement, and more than one arrangement of the cards outside the ranks that make up the straight. Each will be dealt with in turn.

<u>7 cards within the 3 ranks of the straight</u> (no cards outside those ranks): There are 3 cases to consider, 4 cards in one of the straight's ranks, 2 in another and 1 in the third (421), 3 in each of two ranks, and 1 in the other (331), or, 3 in one rank and 2 in each of the others (322). The arrangements and total number of ways these can occur are given in the table:

3-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
421	11	1	6•4•6	1,584
331	11	1	4^3•3	2,112
322	11	1	6^2•4•3	4,752
			Total	**8,448**

<u>6 cards within the 3 ranks of the straight</u> (1 card outside those ranks): There are 3 cases to consider, 4 cards in one of the straight's ranks, 1 in each of the others (411), 3 cards in one rank, 2 in another and 1 in the remaining rank (), or, 2 in each of the ranks (222). In addition there is a separate rank containing 1 card. Arrangements and total number of ways this case can occur are given in the table:

3-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
411 & 1 rank (1)	10C2 (45)	2	4^3•3	17,280
& 1 rank (1)	10C2 (45)	2	4^3•6•6	207,360
222 & 1 rank (1)	10C2 (45)	2	4•6^3	77,760
			Total	**302,400**

<u>5 cards within the 3 ranks of the straight</u> (2 cards outside those ranks): There are 6 cases to consider, 3 cards in one of the straight's ranks, 1 in each of the others

(311), or, 2 in 2 of the ranks and 1 in the other (221). In addition there is a separate rank containing 2 cards, or 2 non-adjacent ranks with 1 card each or a 2-rank straight with 1 card in each rank. Arrangements and total number of ways this case can occur are given in the table:

3-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
311 & 1 rank (2)	10C2 (45)	2	4^3•6•3	103,680
221 & 1 rank (2)	10C2 (45)	2	4•6^3•3	233,280
311 & 2 ranks (1) (1)	9C3 (84)	3	4^5•3	774,144
221 & 2 ranks (1) (1)	9C3 (84)	3	4^3•6^2•3	1,741,824
311 & 2-rank str. (1,1)	9C2 (36)	2	4^5•3	221,184
221 & 2-rank str. (1,1)	9C2 (36)	2	4^3•6^2•3	497,664
			Total	**3,571,776**

<u>4 cards within the 3 ranks of the straight</u> (3 cards outside those ranks): There are 6 cases to consider, 2 cards in one of the straight's ranks, and 1 in each of the others (211), as well as a separate rank containing 3 cards, or 2 non-adjacent ranks with 2 cards in one and one in the other, or 3 non-adjacent, or a 2-rank straight with 3 cards (2,1), a 2-rank straight and a separate rank each with 1 card, or, finally a 3-card straight with one card in each rank. Arrangements and total number of ways this case can occur are given in the table:

3-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
211 & 1 rank (3)	10C2 (45)	2	4^3•6•3	103,680
211 & 2 ranks (1) (2)	9C3 (84)	6	4^3•6^2•3	3,483,648
211 & 3 ranks (1) (1) (1)	8C4 (70)	4	4^5•6•3	5,160,960
211 & 2-rank str. (1,2)	9C2 (36)	2	4^3•6^2•2•3	995,328
211 & 2-rank str. (1,1) & 1 rank (1)	8C3 (56)	6	4^5•6•3	6,193,152
211 & 3-rank str. (111)	8C2 (28)	2	4^5•6•3	1,032,192
			Total	**16,968.960**

<u>3 cards within the 3 ranks of the straight</u> (4 cards outside those ranks): There are 13 cases to consider, a separate rank containing 4 cards, or 2 non-adjacent ranks with 2 cards in each or 3 in one and 1 in the other, or 3 non-adjacent ranks with 2 cards in one rank and 1 in each of the others, or 4 non-adjacent ranks with 1 card in each rank. In addition, there are a 2-rank straight with 2 cards in each rank or 3 in one and 1 in the other, a 2-rank straight with 3 cards (2,1) and a single non-adjacent rank with 1 card, a 2-rank straight with 1 card in each rank and a single rank with 2 cards, a 2-rank straight with 1 card in each rank and a 2 non-adjacent ranks with 1 card each, 2 2-rank straights with 1 card in each rank, a 3-rank straight with 4 cards (211) and a 3-rank straight with 1 card in each rank and 1 separate rank with 1 card. Arrangements and total number of ways this can occur are given in the table:

3-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
111 & 1 rank (4)	10C2 (45)	2	4^3	5,760

111 & 2 ranks (2) (2)	9C3 (84)	3	4^3•6^2	580,608
111 & 2 ranks (3) (1)	9C3 (84)	6	4^5	516,096
111 & 3 ranks (2) (1) (1)	8C4 (70)	12	4^5•6	5,160,960
111 & 4 ranks (1) (1) (1) (1)	7C5 (21)	5	4^7	1,720,320
111 & 2-rank str. (2,2)	9C2 (36)	2	4^3•6^2	165,888
111 & 2-rank str. (1,3)	9C2 (36)	2	4^5•2	147,456
111 & 2-rank str. (1,2) & 1 rank (1)	8C3 (56)	6	4^5•6•2	4.128,768
111 & 2-rank str. (1,1) & 1 rank (2)	8C3 (56)	6	4^5•6	2,064,384
111 & 2-rank str. (1,1) & 2 ranks (1) (1)	7C4 (35)	12	4^7	6,881,280
111 & 2 2-rank str. (1,1) (1,1)	7C3 (35)	3	4^7	1,720,320
111 & 3-rank str. (211)*	8C2 (28)	2	4^5•6•3	—
111 & 3-rank str. (111) & 1 rank (1)	7C4 (35)	3	4^7	1,720,320
			Total	**24,812,160**

*this combination is counted in "4 cards within the 3 ranks….." in the case "211 & 3-rank str. (111)"

The sum of the "Totals" for all 5 variations divided by 52C7 is the probability of the initial layout showing a 3-rank straight as the longest run of adjacent cards,

$$P_{S3} = \frac{8{,}448 + 302{,}400 + 3{,}571{,}776 + 16{,}968{,}960 + 24{,}812{,}160}{52C7} = 0.341322975 \quad (45{,}663{,}744)$$

P_{S4}: For all cases a 4-rank straight is included. There are 4 variations — the 4-rank straight comprising only the 4 cards, the straight comprising 5 cards (i.e. one of the cards is paired up), the straight with 6 cards, and the straight with 7 cards. As for other straights some of these variations have more than one internal arrangement, and more than one arrangement of the cards outside the ranks that make up the straight. Each will be dealt with in turn.

<u>7 cards within the 4 ranks of the straight</u> (no cards outside those ranks): There are 3 cases to consider, 4 cards in one of the straight's ranks, and 1 in each of the others (4111), 3 in one of the ranks, 2 in another, and 1 in each of the remaining 2 ranks (4111), and 2 in each of 3 ranks and 1in the remaining rank (4111). The arrangements and total number of ways this case can occur are given in the table:

4-rank straight	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
4111	10	1	4^3•4	2,560
4111	10	1	4^3•6•12	46,080
4111	10	1	6^3•4•4	34,560
			Total	**83,200**

<u>6 cards within the 4 ranks of the straight</u> (1 card outside those ranks): There are 2 cases to consider, 3 cards in one of the straight's ranks, 1 in each of the others (3111), and 2 in 2 of the ranks and 1 in the other 2 (2211). In addition there is a separate rank containing 1 card. Arrangements and total number of ways this case can occur are given in the table:

4-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
3111 & 1 rank (1)	9C2 (36)	2	4^5•4	294,912
2211 & 1 rank (1)	9C2 (36)	2	4^3•6^2•6	995,328
			Total	**1,290,240**

<u>5 cards within the 4 ranks of the straight</u> (2 cards outside those ranks): There are 3 cases to consider, 3 cards in one of the straight's ranks, 1 in each of the others (311), or, 2 in 2 of the ranks and 1 in the other (221). In addition there is a separate rank containing 2 cards, or 2 non-adjacent ranks with 1 card each or a 2-rank straight with 1 card in each rank. Arrangements and total number of ways this case can occur are given in the table:

4-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
2111 & 1 rank (2)	9C2 (36)	2	4^3•6^2•4	663,552
2111 & 2 ranks (1) (1)	8C3 (56)	3	4^5•6•4	4,128,768
2111 & 2-rank str. (1,1)	8C2 (28)	2	4^5•6•4	1,376,256
			Total	**6,168,576**

<u>4 cards within the 4 ranks of the straight</u> (3 cards outside those ranks): There are 6 cases to consider, 2 cards in one of the straight's ranks, and 1 in each of the others (211), as well as a separate rank containing 3 cards, or 2 non-adjacent ranks with 2 cards in one and one in the other, or 3 non-adjacent, or a 2-rank straight with 3 cards (2,1), a 2-rank straight and a separate rank each with 1 card, or, finally a 3-card straight with one card in each rank. Arrangements and total number of ways this case can occur are given in the table:

4-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
1111 & 1 rank (3)	9C2 (36)	2	4^5	73,728
1111 & 2 ranks (1) (2)	8C3 (56)	6	4^5•6	2,064,384
1111 & 3 ranks (1) (1) (1)	7C4 (35)	4	4^7	2,293,760
1111 & 2-rank str. (1,2)	8C2 (28)	2	4^5•6•2	688,128
1111 & 2-rank str. (1,1) & 1 rank (1)	7C3 (35)	6	4^7	3,440,640
1111 & 3-rank str. (111)	7C2 (21)	2	4^7	688,128
			Total	**9,248,768**

The sum of the "Totals" for all 4 variations divided by 52C7 is the probability of the initial layout showing a 4-rank straight as the longest run of adjacent cards,

$$P_{S4} = \frac{83{,}200 + 1{,}290{,}240 + 6{,}168{,}576 + 9{,}248{,}768}{52C7} = 0.125506142 \quad (16{,}790{,}784)$$

P_{S5}: For all cases a 5-rank straight is included. There are 3 variations — the 5-rank straight comprising only the 5 cards, the straight comprising 6 cards (i.e. one of the cards is paired up), and the straight with 7 cards. As for other straights some of these variations have more than one internal arrangement, and more than one arrangement of the cards outside the ranks that make up the straight. Each will be dealt with in turn.

<u>7 cards within the 5 ranks of the straight</u> (no cards outside those ranks): There are 2 cases to consider, 3 cards in one of the straight's ranks, and 1 in each of the others (31111), 2 in 2 of the ranks, and 1 in each of the others (22111). The arrangements and total number of ways this case can occur are given in the table:

5-rank straight	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
31111	9	1	4^5•5	46,080
22111	9	1	6^2•4^3•10	207,360
			Total	**253,440**

6 cards within the 5 ranks of the straight (1 card outside those ranks): There 1 case to consider, 2 cards in one of the straight's ranks, 1 in each of the others (21111). In addition there is a separate rank containing 1 card. Arrangements and total number of ways this case can occur are given in the table:

5-rank straight & external rank	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
21111 & 1 rank (1)	8C2 (28)	2	4^5•6•5	1,720,320
			Total	**1,720,320**

5 cards within the 5 ranks of the straight (2 cards outside those ranks): There 3 cases to consider, 1 cards in each of the straight's ranks, and either a separate rank containing 2 cards, 2 ranks containing 1 card each, or a 2-rank straight with 1 card in each rank. Arrangements and total number of ways this case can occur are given in the table:

5-rank straight & external ranks	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
11111 & 1 rank (2)	8C2 (28)	2	4^5•6	344.064
11111 & 2 ranks (1) (1)	7C3 (35)	3	4^7	1,720,320
11111 & 2-rank str. (1,1)	7C2 (21)	2	4^7	688,128
			Total	**2,752,512**

The sum of the "Totals" for the 3 variations divided by 52C7 is the probability of the initial layout showing a 5-rank straight as the longest run of adjacent cards,

$$P_{S5} = \frac{253{,}440 + 1{,}720{,}320 + 2{,}752{,}512}{52C7} = 0.035327485 \qquad (4{,}726{,}272)$$

P_{S6}: For all cases a 6-rank straight is included. There are 2 variations — the 6-rank straight comprising only 6 cards, and the straight with all 7 cards. As for other straights some of these variations have more than one internal arrangement, and more than one arrangement of the cards outside the ranks that make up the straight. Each will be dealt with in turn.

7 cards within the 6 ranks of the straight (no cards outside those ranks): There is 1 case to consider, 2 cards in one of the straight's ranks, and 1 in each of the others (211111),. The arrangements and total number of ways this case can occur are given in the table:

6-rank straight	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
211111	8	1	4^5•6•6	294,912
			Total	**294,912**

6 cards within the 6 ranks of the straight (1 card outside those ranks): There is 1 case to consider, 1 in each of the ranks (111111) and a separate rank containing 1 card. Arrangements and total number of ways this case can occur are given in the table:

6-rank straight & external rank	Number of ways to fit in 13 ranks	Number of ways to arrange elements	Number of suit arrangements	Total number of ways
111111 & 1 rank (1)	7C2 (21)	2	4^7	688,128
			Total	**688,128**

The sum of the "Totals" for the 2 variations divided by 52C7 is the probability of the initial layout showing a 6-rank straight as the longest run of adjacent cards,

$$P_{S6} = \frac{294{,}912 + 688{,}128}{52C7} = 0.007347933 \qquad (983{,}040)$$

P_{S7}: There are 7 possible 7-rank straights—A234567 2345678 3456789 …. 78910JQK, and each card can be chosen from any of the 4 suits ($7 \bullet 4^7$ possibilities):

$$P_{S7} = \frac{7 \bullet 4^7}{52C7} = 0.000857259 \qquad (114{,}688)$$

As for rank multiples, the numerators sum to exactly 52C7 so the probabilities sum to unity.

3. COLOUR

P_{C4}: 4 cards of one colour. There are 26C4 ways of choosing the 4 cards from one of the colours and 26C3 ways of choosing the remaining 3 cards from the other colour, and there are 2 colours that the 4 cards can be from:

$$P_{C4} = \frac{26C4 \bullet 26C3 \bullet 2}{52C7} = 0.581083497 \qquad (77{,}740{,}000)$$

P_{C5}: 5 cards of one colour. There are 26C5 ways of choosing the 5 cards from one of the colours and 26C2 ways of choosing the remaining cards from the other colour, and there are 2 colours that the 5 cards can be from:

$$P_{C5} = \frac{26C5 \bullet 26C2 \bullet 2}{52C7} = 0.319595924 \qquad (42{,}757{,}000)$$

P_{C6}: 6 cards of one colour. There are 26C6 ways of choosing the 6 cards from one of the colours and 26C1 ways of choosing the remaining cards from the other colour, and there are 2 colours that the 6 cards can be from:

$$P_{C6} = \frac{26C6 \bullet 26C1 \bullet 2}{52C7} = 0.089486859 \qquad (11{,}971{,}960)$$

P_{C7}: 7 cards of one colour, There are 26C7 ways of choosing the 7 cards from one of the colours and there are 2 colours that the 7 cards can be from:

$$P_{C7} = \frac{26C7 \bullet 2}{52C7} = 0.009833721 \qquad (1{,}315{,}600)$$

The sum of P_{C4} to P_{C7} is 1 exactly (any other possible combinations are included in these 4). It can be noted that the sum of the numerators is a special case of a more general identity $\sum_{i=0}^{n}\binom{M}{i}\binom{N}{n-i} = \binom{M+N}{n}$, where $\binom{j}{k}$ is an alternative notation to jCk.

4, FLUSH

P_{F2}: 2 cards in one suit. There is only one combination that doesn't include 3, 4 or 5 cards in another suit and that is 2221, i.e. 2 cards in one suit, 2 in another suit, 2 in a third suit and 1 card in the 4th suit. There are 13C2 ways to choose 2 cards from one suit, and 3 suits for which this is the case, and 13C1 ways of choosing the remaining card. This makes the product $[13C2]^3 \bullet 13C1$, but there are 4 ways of allowing the single card to be chosen from the 4 suits (the three 2's always take up the other three suits and there is no difference between them in their internal arrangement):

$$P_{F2} = \frac{[13C2]^3 \bullet 13C1 \bullet 4}{52C7} = 0.184451061 \qquad (24{,}676{,}704)$$

P_{F3}: 3 cards in one suit. There are 13C3 ways to choose 3 cards from one suit, and there are 3 further possibilities for the other 4 cards, not allowing 4 cards in one suit: 3 cards in one suit and one in another [13C3•13C1], 2 cards in 2 other suits $[13C2]^2$, and 2 cards in another suit and 1 card each from the other 2 suits $[13C2 \bullet \{13C1\}^2]$. In each of these combinations there are 12 possibilities—for 3310 (the numbers of cards in each suit) there are 4 possibilities for the 1, 3 left for the 0, or vice versa, and the 3's take up the remaining 2 suits; for the 3220 the same argument holds; and for the 3211 again the same holds:

$$P_{F3} = \frac{13C3 \bullet (13C3 \bullet 13C1 + [13C2]^2 + 13C2 \bullet [13C1]^2) \bullet 4 \bullet 3}{52C7} = 0.589612792 \ (78{,}881{,}088)$$

P_{F4}: 4 cards in one suit. There are 13C4 ways to choose 4 cards from one suit, 4 suits, and 39C3 ways to choose the remaining 3 cards from the other 3 suits:

$$P_{F4} = \frac{13C4 \bullet 39C3 \bullet 4}{52C7} = 0.195370378 \qquad (26{,}137{,}540)$$

P_{F5}: 5 cards in one suit. There are 13C5 ways to choose 5 cards from one suit, 4 suits, and 39C2 ways to choose the remaining 2 cards from the other 3 suits:

$$P_{F5} = \frac{13C5 \bullet 39C2 \bullet 4}{52C7} = 0.028513515 \qquad (3{,}814{,}668)$$

P_{F6}: 6 cards in one suit. There are 13C6 ways to choose 6 cards from one suit, 4 suits, and 39C1 ways to choose the remaining card from the other 3 suits:

$$P_{F6} = \frac{13C6 \bullet 39C1 \bullet 4}{52C7} = 0.002000948 \qquad (267{,}696)$$

P_{F7}: 7 cards in one suit. There are 13C7 ways to choose 7 cards from one suit, and 4 suits:

$$P_{F7} = \frac{13C7 \bullet 4}{52C7} = 0.000051306 \qquad (6{,}864)$$

The numerators sum to exactly 52C7 so the probabilities sum to unity.

D. Comparisons with Growly Solitaire

1. MULTIPLES WITHIN RANKS

Comparison measurements to the 11 probabilities discussed above were obtained using Growly Solitaire. "New Game" was repeatedly accessed to produce new layouts and the number of cases occurring in the initial turned-up layout cards for each of the multiple's combinations was counted and divided by the total number of trials (layouts). The following table contains the results:

	P1	P2	P22	P222	P3	P32	P322	P33	P4	P42	P43
# of trials	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
# of counts	223	469	231	11	36	28	1	0	1	0	0
Meas. Prob.	0.223	0.469	0.231	0.0110	0.0360	0.0280	0.0010	0.000	0.00100	0.000	0.000
Theor. Prob.	0.210	0.473	0.222	0.01847	0.0493	0.0246	0.000924	0.000410	0.00137	0.0003078	0.00000466
σ	0.0129	0.0158	0.0131	0.00426	0.00684	0.00490	0.000961	0.000641	0.00117	0.000555	0.0000683
# of σ's	0.997	0.243	0.712	1.754	1.937	0.688	*Too few*	*Too few*	*Too few*	*Too few*	*Too few*

Comments:

(a) The standard deviation σ is computed using the theoretical probability (p) but pertains to the measured probability which is based on a histogram. [$\sigma = \sqrt{p(1-p)/(\#\ of\ trials)}$]

(b) There are 2 cases in which the difference between the measured probability and the theoretical probability is greater than 1σ, namely for P_{222} and P_3, and for all 6 measurable cases the difference is less than 2σ.

(c) There are 5 cases in which the number of measurement counts is less than 5 and for which the expected number of counts is also less than 5, so they have been designated as *"Too few*", and for purposes of calculating a χ^2-value they have been pooled with the next smallest case, P222. The resulting value of χ^2 is 8.59 with 5 degrees of freedom and has a significance level of 13%.

2. STRAIGHTS

Again, the comparison measurements are summarized in the following table:

	S1	S2	S3	S4	S5	S6	S7	Sum
# of trials	2000	2000	2000	2000	2000	2000	2000	
# of counts	94	924	669	228	73	12	0	
Measured Prob.	0.0470	0.462	0.3345	0.114	0.0365	0.0060	0.000	1
Theoretical Prob.	0.04873	0.4413	0.3413	0.1255	0.03533	0.007348	0.0008573	1

σ	0.00480	0.0111	0.0106	0.00741	0.00413	0.00191	0.00065	
# of σ's	0.36	1.87	0.64	1.55	0.28	0.71	*Too few*	

Comments:

(a) The standard deviation σ is computed as described for rank multiples.

(b) In all cases the difference between the measured probability and the theoretical probability is less than 2σ, and in all cases but one ($S2$) it is less than 1.6σ.

(c) There is 1 case, S7, in which the number of expected counts is less than 5 so it has been designated as *"Too few"* and has been pooled with the next smallest, S6, to calculate a χ^2-value of 5.71 (with 5 degrees of freedom) for which the level of significance is 34%.

3. COLOUR

The comparisons are as follows:

	C4	C5	C6	C7	Sum
# of trials	500	500	500	500	
# of counts	299	159	39	3	
Meas. Prob.	0.598	0.318	0.078	0.006	1
Theoretical Prob.	0.5811	0.3196	0.0895	0.00983	1
σ	0.0221	0.0209	0.0128	0.0044	
# of σ's	0.77	0.08	0.90	*Too Few*	

Comments:

(a) In all cases the difference between the measured probability and the theoretical probability is less than 1σ.

(b) There is 1 case in which the number of measurement counts is *"Too few"* ($C7$), and it has been pooled with the next smallest, C6, to calculate a χ^2-value of 1.43 (with 2 degrees of freedom) for which the level of significance is 49%.

4. FLUSH

The comparisons are as follows:

	F2	F3	F4	F5	F6	F7	Sum
# of trials	500	500	500	500	500	500	
# of counts	95	299	95	11	0	0	
Meas. Prob.	0.19	0.598	0.19	0.022	0	0	1
Theoretical Prob.	0.1845	0.5896	0.1954	0.02851	0.00200	0.00005	1
σ	0.0173	0.0220	0.0177	0.00744	0.0020	0.0001	
# of σ's	0.32	0.38	0.30	0.87	*Too few*	*Too few*	

Comments:

(a) In all cases the difference between the measured probability and the theoretical probability is less than 1σ.

(b) There are 2 cases, $F6$ and $F7$, in which the number of measurement counts are *"Too few"* and they have been pooled with the next smallest, F5, to calculate a χ^2-value of 1.404 (with 3 degrees of freedom) for which the level of significance is 70%.

All of the comparisons provide a confirmation that the Growly Solitaire results for these cases are not significantly different from the theoretical values, and because of the value of χ^2 for rank multiples (comment 3), this is conditional on using a 90% confidence level or greater.

One further note can be made concerning the comparison results. From the definition of σ (rank multiples, comment 1) it can be seen that the difference between the measured and expected probability values should decrease as *(# of trials)*$^{-1/2}$ so that the convergence rate toward the mathematical expectation is quite slow. To obtain an extra decimal place in accuracy would require 100 times as many trials. Results[2] gathered by digital simulation to compare the expected and observed frequencies of only one colour showing in the layout cards (including the first turned up card from the deck) indicate that approximately 10^6 trials are needed before the observed and expected values are the same to within 2 significant figures, and 10^8 for 3 figures. The expected probability in this case is $2 \bullet 26C8/52C8$ which is 0.0041520….

E. Summary and Conclusions

There are two conclusions, one regarding the mathematical probabilities of the 28 combinations that have been determined here, and the second regarding the closeness of the Growly Solitaire results to these probabilities. The values of the mathematical probabilities are summarized in the following tables:

	Multiples Within Ranks	**Count out of 52C7**	**Probability**	**Average Number of Trials to a Repeat**
P_1	All cards with different ranks	28,114,944	0.210150887	4.8
P_2	One pair	63,258,624	0.472839497	2.1 (1.4 any pair)
P_{22}	Two pairs	29,652,480	0.221643514	4.5
P_{222}	Three pairs	2,471,040	0.018470293	63
P_3	One 3-of-a-kind (triple)	6,589,440	0.049254114	20 (13 any triple)
P_{32}	One 3-of-a-kind and a pair (full house)	3,294,720	0.024627057	41
P_{322}	One 3-of-a-kind and 2 pairs	123,552	0.000923515	1,082
P_{33}	Two 3's-of-a-kind	54,912	0.000410451	2,436
P_4	One 4-of-a-kind (quadruple)	183,040	0.001368170	731 (595 any quad.)
P_{42}	One 4-of-a-kind and one pair	41,184	0.000307838	3,249
P_{43}	One 4-of-a-kind and one 3-of-a-kind	624	0.000004664	214,398
	TOTAL (52C7)	**133,784,560**	**1**	

	Straights	**Count out of 52C7**	**Probability**	**Average Number of Trials to a Repeat**
P_{S1}	all cards from non-adjacent ranks	6,471,184	0.048370186	21
P_{S2}	2-rank straights	59,034,848	0.441268021	2.3
P_{S3}	3-rank straights	45,663,744	0.341322975	3.9
P_{S4}	4-rank straights	16,790,784	0.125506142	8.0
P_{S5}	5-rank straights	4,726,272	0.035327485	28
P_{S6}	6-rank straights	983,040	0.007347933	136
P_{S7}	7-rank straights	114,688	0.000857259	1,167
	TOTAL (52C7)	**133,784,560**	**1**	

Colours		Count out of 52C7	Probability	Average Number of Trials to a Repeat
P_{C4}	4 cards of one colour	77,740,000	0.581083497	1.7
P_{C5}	5 cards of one colour	42,757,000	0.319595924	3.1
P_{C6}	6 cards of one colour	11,971,960	0.089486859	11
P_{C7}	7 cards of one colour	1,315,600	0.009833721	102
	TOTAL (52C7)	**133,784,560**	**1**	

Flushes		Count out of 52C7	Probability	Average Number of Trials to a Repeat
P_{F2}	maximum of 2 cards in one suit	24,676,704	0.184451061	5.4
P_{F3}	maximum of 3 cards in one suit	78,881,088	0.589612792	1.7
P_{F4}	4 cards in one suit	26,137,540	0.195370378	5.1
P_{F5}	5 cards in one suit	3,814,668	0.028513515	35
P_{F6}	6 cards in one suit	267,696	0.002000948	500
P_{F7}	7 cards in one suit	6,864	0.000051306	19,489
	TOTAL (52C7)	**133,784,560**	**1**	

There are comparable probabilities available in the public domain which show identical values for identical cases. In particular, the probabilities for some 7-card poker hands[3,4,5] are directly obtainable from probabilities for rank multiples, flush and straight:

Item	Present Analysis	Present Analysis Count	Poker Hand Count[3,4,5]
Full House	$P_{32} + P_{322} + P_{33}$	3,473,184	3,473,184
4-of-a-Kind	$P_4 + P_{42} + P_{43}$	224,848	224,848
5-card Flush	$P_{F5} + P_{F6} + P_{F7}$	4,089,228	4,089,228*
5-card Straight	$P_{S5} + P_{S6} + P_{S7}$	5,824,000	5,824,000**

* includes straight flush and royal flush — the present probabilities include A,2,3,4,5 and 10,J,Q,K,A because rank is not important in the present calculation of flush probabilities, however both of these are part of the straight flushes that have not been included in the flush probabilities for poker (4,047,644) so they both have to be added back in (37,260 is the straight flush count for Ace up to King, and 4,324 is the count for royal flushes).

** from Alspach[5], with modifications so flushes are not removed and Ace is fixed to be low.

If for any particular computer version of Klondike solitaire, or any card-based game, the frequencies of appearance of any of the patterns examined here are significantly different from those given in the above tables, it may well be an indication of some inadequacy in the internal workings of the computer game or of the shuffling process for the card-based games. As examples, if 3-of-a-kind appear in the initial layout significantly more often than 1 in every 13 games on an average, i.e. 1/probability, (this includes all ways a 3-of-a-kind can appear, P_3, P_{32}, P_{322} and P_{33}), or if 5 cards in the same suit occur more often than once in about 35 trials, or a 4-rank straight appears more often than once in every 8 games or so in the initial layout, perhaps the games are not operating as closely to the unbiased and fully-shuffled condition as they might be. On the other hand, if the initial layouts are in accord with these appearance rates, there is no reason to suspect that the game is operating improperly.

The second conclusion is the closeness of the Growly Solitaire trial results to the mathematical probabilities, implying that the Growly Solitaire application is not shown to have statistical bias, nor is it shown to exhibit more than expected deviations from true randomness. From the tables in which comparisons of theoretical and measured probabilities are provided, out of the total of 19 actual comparisons (6 for rank multiples, 6 for straights, 3 for colour and 4 for flush) only 4 comparisons show occurrence rates that deviate from the expected values by more than 1σ — and all comparisons show differences that are less than 2σ. The χ^2 values for each category (rank multiples, straights, colours and flushes) are all small enough that the significance levels are > 10% and as such are not considered to show meaningful deviations from the mathematically expected values.

On this basis, the Growly Solitaire application can be considered to provide a reliable version of Klondike Solitaire.

Finally, given that the convergence rate is ~*(# of trials)*$^{-1/2}$, improving the convergence would likely require some digital assistance rather than performing each layout manually with a computer solitaire game (as was done here).

F. References

1 .*http://www.growlybird.com/solitaire/index.html*, 2017.
2. Turnbull, Bill, *http://www.roziturnbull.com/bill/Solitaire/redblack.html,* July 2011.
3. *https://en.wikipedia.org/wiki/Poker_probability* In this, 'count' is referred to as 'frequency', and their counts for straights allow Ace to be high or low. September 2017.
4. Butler, Bill, *http://www.durangobill.com/Poker_Probabilities_7_Cards.html* In this, 'count' is referred to as 'Nbr of Hands', and as in reference 3, their counts for straights allow Ace to be high or low.
5. Alspach, B, *http://people.math.sfu.ca/~alspach/comp20/*, a very good summary of poker probabilities with a full description of the mathematical calculations. Again, Ace is high or low. January 2000.